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ANGULAR MOMENTUM DEPENDENT POTENTIALS IN NUCLEON-NUCLEON SCATTERING

BY

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Synopsis

We investigate the general properties of the most general local, angular momentum dependent nucleon-nucleon potential. We generalise the arguments of a previous paper on angular momentum dependent potentials, to take into account the complications which arise because of the spin of the nucleons. As an example we consider a scalar boson exchange potential, and show that the conventional approximation methods for obtaining local potentials from field theory are completely misleading in this case.

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1. Introduction

In a previous paper[1] (referred to as I hereafter) we have discussed the concept of a local, angular momentum dependent potential, and its relevance to the equivalence problem, that is, the problem of obtaining a local potential which is equivalent to a given non-local potential as far as phase shifts are concerned. We established in I that it is in principle possible to construct a local angular momentum dependent potential which is equivalent to a given non-local one. The question whether a potential is non-local or local in coordinate space depends on the off shell behaviour of the potential in momentum space. In I it was explicitly shown that in perturbation theory it is always possible to choose the off shell continuation of the potential so that the resulting coordinate space potential becomes local and angular momentum dependent, since this merely corresponds to a rearrangement of the (infinite number of) equations connecting the potentials V_{2n} to the T-matrices T_{2n} of various orders. In a practical calculation, when the potential is constructed only up to some finite (and small) order, the off shell continuation chosen for the potential will affect the resulting phase shifts. On the basis of numerical calculations performed in I for a single particle exchange potential, we may assert that the phase shift obtained with a potential which is chosen to be local and angular momentum dependent does not differ very much from a phase shift obtained with a potential which differs slightly off the energy shell from the first potential. From the practical point of view, we may therefore consider the method of constructing a local, angular momentum dependent potential as a method by which a given non-local potential can be approximated by a local one. An approximation of this kind can in general be expected to be superior to the "static" and "adiabatic" approximations which have hitherto been used.

The purpose of the present paper is to generalise the arguments given in I for interactions between spinless scalar particles to the case of interactions between nucleons in which various complications occur due to the spin of the nucleons. However, in this paper we restrict the detailed discus-

sion to only those points which cannot immediately be inferred from the discussion given in I.

Section 2 contains a discussion of the general form of a potential. In section 3 we evaluate the "spin-angle"-matrix elements of the potential in the momentum representation, which are needed in order to obtain the partial wave integral equations which connect the angular momentum dependent coordinate space potentials to the potential in momentum representation.

In section 4 we solve these integral equations, and examine briefly some of the properties of the resulting coordinate space potentials.

Section 5 contains a discussion of the scalar boson exchange potential which we use as an illustration, and in section 6 we give some concluding remarks.

2. The general form of the potential

As is shown in an article by J. Goto and S. Machida[2]), the most general form of a potential between two spin one half nucleons, which fulfils natural invariance requirements, *i.e.* invariance with respect to coordinate space translation, Galilei transformation, the exchange of the two nucleons, rotation of space coordinates, space reflections, time reversal and Hermiticity of the potential, is, in momentum space,

$$\left.\begin{aligned} V(\sigma_1,\sigma_2,\boldsymbol{q},\boldsymbol{p}) \;=\; & V_0 + V_1 i(\boldsymbol{q}\times\boldsymbol{p})\cdot\boldsymbol{S} + V_2\sigma_1\cdot\boldsymbol{q}\sigma_2\cdot\boldsymbol{q} + V_3\sigma_1\cdot\sigma_2 \\ & + V_4\sigma_1\cdot(\boldsymbol{q}\times\boldsymbol{p})\sigma_2\cdot(\boldsymbol{q}\times\boldsymbol{p}) + V_5\sigma_1\cdot\boldsymbol{p}\sigma_2\cdot\boldsymbol{p}. \end{aligned}\right\} \tag{2.1}$$

The quantities $\boldsymbol{q}$ and $\boldsymbol{p}$ are given, in terms of the centre of mass (c.m.) momenta (Fig. 1), by

$$\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}', \qquad \boldsymbol{p} = \tfrac{1}{2}(\boldsymbol{k} + \boldsymbol{k}'). \tag{2.2}$$

The six functions V_i in (2.1) are real functions of the three independent scalars that can be formed from $\boldsymbol{q}$ and $\boldsymbol{p}$,

$$V_i = V_i(\boldsymbol{q}^2,\boldsymbol{p}^2,(\boldsymbol{q}\times\boldsymbol{p})^2). \tag{2.3}$$

Finally, σ_1 and σ_2 in (2.1) denote the spin operators for the two nucleons, and $\boldsymbol{S} = 1/2(\sigma_1 + \sigma_2)$.

When charge independence is assumed, then each function V_i can be expressed as a sum of a **1**-term and $\tau_1\cdot\tau_2$-term in iso-space, where the τ are the iso-spin operators. In what follows we shall omit the iso-spin factors, which are unessential in the discussion.

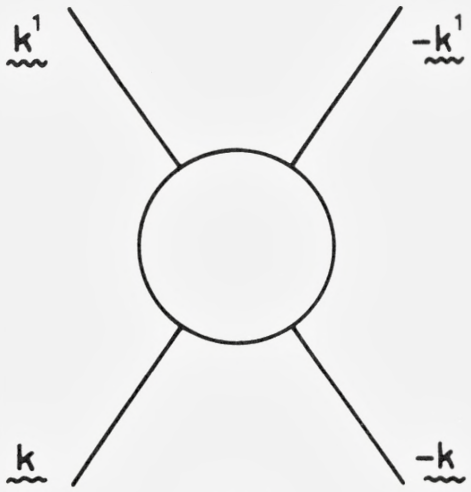

Fig. 1.

When (2.1) is transformed to coordinate space, one obtains in general a completely non-local potential.

However, in certain circumstances the resulting coordinate space potential becomes neither strictly local nor non-local, but becomes a function of r and $\boldsymbol{L}^2$, where $\boldsymbol{L}$ is the angular momentum operator. The problem of determining the form of such a potential in the two-nucleon case has been considered by S. Okubo and R. E. Marshak[3]. Their result, which is essentially based on invariance arguments, is

$$G(\sigma_1, \sigma_2, r, \boldsymbol{L}^2) = G_0 + G_1 \boldsymbol{L} \cdot \boldsymbol{S} + \tfrac{1}{2}[G_2, S_{12}]_+ + G_3 \sigma_1 \cdot \sigma_2 + G_4 L_{12}, \qquad (2.4)$$

where

$$S_{12} = 3r^{-2} \sigma_1 \cdot \boldsymbol{r} \sigma_2 \cdot \boldsymbol{r} - \sigma_1 \cdot \sigma_2, \qquad (2.5)$$

$$L_{12} = 3\sigma_1 \cdot \boldsymbol{L} \sigma_2 \cdot \boldsymbol{L} - \sigma_1 \cdot \sigma_2 \boldsymbol{L}^2. \qquad (2.6)$$

The five functions G_i in (2.4) are functions of r and $\boldsymbol{L}^2$ only. Actually (2.4) differs slightly from the expression given by Okubo and Marshak, as we have symmetrised the tensor force term, and, for reasons of convenience, changed the definition of the quadratic spin orbit operator L_{12}.

As pointed out by Okubo and Marshak, a potential of the form (2.4) is the most general local potential which can be obtained as a solution to the inverse problem of scattering.

Let us now suppose that the function V in (2.1) is properly restricted so that it has a coordinate space representation of the form (2.4). If this is the case, then we have the relation between V and G

$$V(\sigma_1, \sigma_2, \boldsymbol{q}, \boldsymbol{p}) = \langle \boldsymbol{k}' \mid G(\sigma_1, \sigma_2, r, \boldsymbol{L}^2) \mid \boldsymbol{k} \rangle \tag{2.7}$$

where

$$\langle \boldsymbol{k}' \mid G(\sigma_1, \sigma_2, r, \boldsymbol{L}^2) \mid \boldsymbol{k} \rangle = \int d^3\boldsymbol{r} e^{-i\boldsymbol{k}' \cdot \boldsymbol{r}} G(\sigma_1, \sigma_2, r, \boldsymbol{L}^2) e^{i\boldsymbol{k} \cdot \boldsymbol{r}}. \tag{2.8}$$

If the relation (2.7) holds between V and G, then it holds separately for the spin-independent, spin-linear and spin-bilinear parts of V and G respectively. We thus obtain from (2.7),

$$V_0 = \langle \boldsymbol{k}' \mid G_0 \mid \boldsymbol{k} \rangle \tag{2.9}$$

$$V_1 i(\boldsymbol{q} \times \boldsymbol{p}) \cdot \boldsymbol{S} = \langle \boldsymbol{k}' \mid G_1 \boldsymbol{L} \cdot \boldsymbol{S} \mid \boldsymbol{k} \rangle \tag{2.10}$$

$$\sum_{i=2}^{5} V_i \Omega_i = \langle \boldsymbol{k}' \mid \left\{ \tfrac{1}{2}[G_2 S_{12}]_+ + G_3 \sigma_1 \cdot \sigma_2 + G_4 L_{12} \right\} \mid \boldsymbol{k} \rangle, \tag{2.11}$$

where we have denoted by Ω_i $(i = 2, \ldots, 5)$ the four spin-bilinear expressions in (2.1).

In the first place it is clear that the relations (2.9)–(2.11) are well defined only if the functions $G_i(r, \boldsymbol{L}^2)$ satisfy certain conditions. In I we have analysed these conditions in detail for the case of a spin independent potential, which corresponds to eq. (2.9). We may summarize these conditions as follows. We require that each function $G_i\big(r, L(L+1)\big)$ for fixed $r > 0$ is an entire function of $L(L+1)$, which is bounded by a finite power of $L(L+1)$ for non-negative integral values of L. Then we require the existence of the integrals

$$\int_0^\infty dr r^2 \left| G_i\big(r, L(L+1)\big) \right| \tag{2.12}$$

for every fixed L. In addition to (2.12) we have also to require the existence of additional absolute moments of each function $G_i\big(r, L(L+1)\big)$, but on this point we refer to I for details.

Our next concern will be to invert the equations (2.9)–(2.11), that is, express the functions G_i as integral transforms of the functions V_j. At this point it is convenient to take the partial wave projection of the equations (2.9)–(2.11).

We define the partial wave projection or "spin-angle" matrix element of a given function $V(\sigma_1,\sigma_2,\boldsymbol{q},\boldsymbol{p})$ as follows:

$$V_{LL'} \equiv \langle L'|V|L\rangle = \int d\Omega_k d\Omega_{k'} Y^{M*}_{JL'S}(\Omega_{k'}) V Y^{M}_{JLS}(\Omega_k) \tag{2.13}$$

where

$$Y^{M}_{JLS}(\Omega) = \sum_{m_L,m_S} C_{LS}(J,M;m_L,m_S) Y_{L,m_L}(\Omega)\chi_S^{m_S} \tag{2.14}$$

in standard notation.

3. Matrix elements

The matrix elements of the six potential terms in (2.1) have been evaluated by GOTO and MACHIDA (Ref. (2)), but for completeness these matrix elements are also included in this paper in a more compact notation. The matrix elements we have calculated coincide with those given by GOTO and MACHIDA, except in the cases of the linear and quadratic spin orbit potential, where there is a discrepancy. We also calculate the matrix elements of the five terms in $\langle \boldsymbol{k}'|G|\boldsymbol{k}\rangle$. The evaluation of the matrix elements defined by eq. (2.13) is in principle quite straightforward, although much tedious Clebsch-Gordon algebra is required in the calculation. To simplify the notation we introduce the following abbreviations:

$$\left.\begin{aligned} A^{i(m)}_L \equiv A^{i(m)}_L(k,k') &= (1+3\delta_{i5})^{-1} \int_{-1}^{+1} dx x^m P_L(x) V_i \\ A^{i}_L(k,k') &\equiv A^{i(0)}_L(k,k'), \end{aligned}\right\} \tag{3.1}$$

and

$$G^i_L(k,k') = (2\pi)^3 (kk')^{-\frac{1}{2}} \int_0^\infty dr r G_i\big(r,L(L+1)\big) J_{L+\frac{1}{2}}(kr) J_{L+\frac{1}{2}}(k'r) \tag{3.2}$$

Here $J_\nu(z)$ is the Bessel function of the first kind, $P_L(x)$ is the Legendre polynominal and x is the cosine of the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$. Instead of *e.g.* $\langle L',\boldsymbol{k}'|G_0|\boldsymbol{k},L\rangle$ we write $\langle L'|G_0|L\rangle$ to simplify the notation. Because of the symmetry, invariance and Hermiticity requirements which we assume the potential to satisfy there will be five independent matrix elements; namely one for $S = 0$, where we have $L' = L = J$, and for $S = 1$ three matrix elements between states of equal L, where L takes the values $J-1$, J and $J+1$ respectively, and one non-diagonal element with $L' = J-1$, $L = J+1$.

For the spin-independent and spin-linear potential we readily obtain

$$\langle L'|V_0|L\rangle = 2\pi\delta_{LL'}F_L^0(k,k'), \quad F_L^0(k,k') = A_L^0, \tag{3.3}$$

$$\langle L'|G_0|L\rangle = \delta_{LL'}G_L^0(k,k'), \tag{3.4}$$

and

$$\left.\begin{aligned} &\langle L'|V_1 i(\boldsymbol{q}\times\boldsymbol{p})\cdot\boldsymbol{S}|L\rangle = \\ &\tfrac{1}{2}[J(J+1) - L(L+1) - S(S+1)]\delta_{LL'}2\pi F_L^1(k,k'), \\ &F_L^1(k,k') = kk'\left(\frac{A_{L+1}^1 - A_{L-1}^1}{2L+1}\right), \end{aligned}\right\} \tag{3.5}$$

$$\langle L'|G_1\boldsymbol{L}\cdot\boldsymbol{S}|L\rangle = \delta_{LL'}\tfrac{1}{2}[J(J+1) - L(L+1) - S(S+1)]G_L^1(k,k'). \tag{3.6}$$

The evaluation of the matrix elements of the bilinear terms requires an appreciable amount of calculation, the result of which can be expressed as follows. The four matrix elements between states of equal L of any of the functions $V_i\Omega_i (i = 2,\ldots,5)$ can be written as

$$\langle L|V_i\Omega_i|L\rangle = 2\pi F_L^{i(SP)}(k,k')\langle\sigma_1\cdot\sigma_2\rangle + 2\pi F_L^{i(QT)}(k,k')\langle S_{12}\rangle, \tag{3.7}$$

where

$$\langle\sigma_1\cdot\sigma_2\rangle = 2S(S+1) - 3, \tag{3.8}$$

and

$$\langle S_{12}\rangle = \left\{\begin{aligned} &0, && \text{for} \quad S = 0, \\ &-\frac{2(J-1)}{2J+1}, && \text{for} \quad S = 1, \quad L = J-1, \\ &2, && \text{for} \quad S = 1, \quad L = J, \\ &-\frac{2(J+2)}{2J+1}, && \text{for} \quad S = 1, \quad L = J+1. \end{aligned}\right. \tag{3.9}$$

We obtain

$$F_L^{2(SP)}(k,k') = \tfrac{1}{3}[(k^2 + k'^2)A_L^2 - 2kk'A_L^{2(1)}], \tag{3.10}$$

$$F_L^{2(QT)}(k,k') = \tfrac{1}{3}\left[(k^2 + k'^2)A_L^2 - 2kk'A_L^{2(1)} + 3kk'\left(\frac{A_{L+1}^2 - A_{L-1}^2}{2L+1}\right)\right], \tag{3.11}$$

$$F_L^{3(SP)}(k,k') = A_L^3, \tag{3.12}$$

$$F_L^{3(QT)}(k,k') = 0, \tag{3.13}$$

$$F_L^{4(SP)}(k,k') = \tfrac{1}{3}k^2k'^2[A_L^4 - A_L^{4(2)}], \tag{3.14}$$

$$F_L^{4(QT)}(k,k') = \tfrac{2}{3}k^2k'^2\left[A_L^{4(2)} - A_L^4 + \tfrac{3}{2}\left(\frac{A_{L-1}^{4(1)} - A_{L+1}^{4(1)}}{2L+1}\right)\right], \tag{3.15}$$

and

$$F_L^{5(SP)}(k,k') = \tfrac{1}{3}[(k^2 + k'^2)A_L^5 + 2kk'A_L^{5(1)}], \tag{3.16}$$

$$F_L^{5(QT)}(k,k') = \tfrac{1}{3}\left[(k^2 + k'^2)A_L^5 + 2kk'A_L^{5(1)} - \tfrac{3}{2}\left(\frac{A_{L+1}^5 - A_{L-1}^5}{2L+1}\right)\right]. \tag{3.17}$$

The non-diagonal element can be written as

$$\langle J+1|V_i\Omega_i|J-1\rangle = \frac{6\pi(J(J+1))^{\frac{1}{2}}}{2J+1}F_J^{i(T)}(k,k'), \tag{3.18}$$

where

$$F_L^{2(T)}(k,k') = \tfrac{2}{3}(k^2A_{L-1}^2 + k'^2A_{L+1}^2 - 2kk'A_L^2), \tag{3.19}$$

$$F_L^{3(T)}(k,k') = 0, \tag{3.20}$$

$$F_L^{4(T)}(k,k') = \tfrac{2}{3}k^2k'^2[2A_L^{4(1)} - A_{L-1}^4 - A_{L+1}^4], \tag{3.21}$$

$$F_L^{5(T)}(k,k') = \tfrac{2}{3}[k'^2A_{L-1}^5 + k^2A_{L+1}^5 + 2kk'A_L^5]. \tag{3.22}$$

Hence, if

$$V = \sum_{i=2}^{5} V_i\Omega_i, \tag{3.23}$$

then

$$\langle L|V|L\rangle = 2\pi F_L^{(SP)}(k,k')\langle\sigma_1\cdot\sigma_2\rangle + 2\pi F_L^{(QT)}(k,k')\langle S_{12}\rangle, \tag{3.24}$$

and

$$\langle J+1|V|J-1\rangle = \frac{6\pi(J(J+1))^{\frac{1}{2}}}{2J+1}F_J^{(T)}(k,k'), \tag{3.25}$$

where

$$\left.\begin{aligned} F_L^{(SP)}(k,k') &= \sum_{i=2}^{5} F_L^{i(SP)}(k,k'), \\ F_L^{(QT)}(k,k') &= \sum_{i=2}^{5} F_L^{i(QT)}(k,k'), \end{aligned}\right\} \tag{3.26}$$

and

$$F_L^{(T)}(k,k') = \sum_{i=2}^{5} F_L^{i(T)}(k,k'). \tag{3.27}$$

For the matrix elements of the bilinear terms in G we obtain

$$\langle L|\tfrac{1}{2}[G_2, S_{12}]_+|L\rangle = G_L^2(k,k')\langle S_{12}\rangle, \tag{3.28}$$

and

$$\left.\begin{aligned} &\langle J+1|\tfrac{1}{2}[G_2,S_{12}]_+|J-1\rangle = \\ &-\frac{6(J(J+1))^{\frac{1}{2}}}{2J+1}\tfrac{1}{2}[G^2_{J-1,J+1}(k,k')+G^2_{J+1,J-1}(k,k')], \end{aligned}\right\} \tag{3.29}$$

where

$$\left.\begin{aligned} &G^2_{L-1,L+1}(k,k')+G^2_{L+1,L-1}(k,k') = \\ &(2\pi)^3(kk')^{-\frac{1}{2}}\int_0^\infty drr[G_2(r,(L-1)L)+G_2(r,(L+1)(L+2))] \\ &J_{L-\frac{1}{2}}(kr)J_{L+\frac{3}{2}}(k'r), \end{aligned}\right\} \tag{3.30}$$

$$\langle L'|G_3\sigma_1\cdot\sigma_2|L\rangle = \delta_{LL'}(2S(S+1)-3)G^3_L(k,k'), \tag{3.31}$$

and

$$\langle L'|G_4L_{12}|L\rangle = \delta_{LL'}\langle L_{12}\rangle G^4_L(k,k'), \tag{3.32}$$

where

$$\langle L_{12}\rangle = \left\{\begin{array}{llll} 0, & \text{for} & S=0, & \\ (J-1)(2J-3), & \text{for} & S=1, & L=J-1, \\ -(2J-1)(2J+3), & \text{for} & S=1, & L=J, \\ (J+2)(2J+5), & \text{for} & S=1, & L=J+1. \end{array}\right. \tag{3.33}$$

We observe the relation between $\langle S_{12}\rangle$ and $\langle L_{12}\rangle$

$$-2\langle L_{12}\rangle = (2L-1)(2L+3)\langle S_{12}\rangle. \tag{3.34}$$

4. The integral equations and their solutions

We can now write down the integral equations which correspond to the partial wave projections of the equations (2.9), (2.10) and (2.11) respectively. We introduce the abbreviation

$$M_{L,L'}(k,k';r) = J_{L+\frac{1}{2}}(kr)J_{L'+\frac{1}{2}}(k'r). \tag{4.1}$$

From eqs. (3.3) and (3.4) we obtain

$$(kk')^{\frac{1}{2}}F^0_L(k,k') = 4\pi^2\int_0^\infty drrG_0(r,L(L+1))M_{L,L}(k,k';r), \tag{4.2}$$

and from eqs. (3.5) and (3.6)

$$(k'k)^{\frac{1}{2}}F^1_L(k,k') = 4\pi^2\int_0^\infty drrG_1(r,L(L+1))M_{L,L}(k,k';r). \tag{4.3}$$

Let us then consider the bilinear terms in G

$$\tfrac{1}{2}[G_2, S_{12}]_+ + G_3\sigma_1 \cdot \sigma_2 + G_4 L_{12}. \tag{4.4}$$

From (3.28), (3.31) and (3.32) we obtain the expression for the matrix element of (4.4), between states of equal L,

$$G_L^2(k,k')\langle S_{12}\rangle + G_2^3(k,k')\langle \sigma_1 \cdot \sigma_2\rangle + G_L^4(k,k')\langle L_{12}\rangle. \tag{4.5}$$

A comparison of (3.24) and (4.5) gives directly

$$(kk')^{\frac{1}{2}} F_L^{(SP)}(k,k') = 4\pi^2 \int_0^\infty dr r G_3(r, L(L+1)) M_{L,L}(k,k';r), \tag{4.6}$$

and, using eq. (3.34),

$$\left.\begin{aligned} (kk')^{\frac{1}{2}} F_L^{(QT)}(k,k') &= 4\pi^2 \int_0^\infty dr r G_2(r, L(L+1)) M_{L,L}(k,k';r) \\ -\tfrac{1}{2}(2L-1)(2L+3)4\pi^2 &\int_0^\infty dr r G_4(r, L(L+1)) M_{L,L}(k,k';r). \end{aligned}\right\} \tag{4.7}$$

For the non-diagonal matrix element we obtain, from eqs. (3.25) and (3.29),

$$\left.\begin{aligned} (kk')^{\frac{1}{2}} F_L^{(T)}(k,k') &= \\ -4\pi^2 \int_0^\infty dr r [G_2(r,(L-1)L) + G_2(r,(L+1)(L+2))] &M_{L-1,L+1}(k,k';r) \end{aligned}\right\} \tag{4.8}$$

The integral equations we have obtained, eqs. (4.2), (4.3) and eqs. (4.6)–(4.8) are the consequences of the assumption that the function $V(\sigma_1,\sigma_2,\boldsymbol{q},\boldsymbol{p})$ is represented by a function $G(\sigma_1,\sigma_2,r,\boldsymbol{L}^2)$ in coordinate space.

The equation corresponding to (4.2) has been investigated in I, where it was shown that whenever the equation has a solution for $G_0(r,L(L+1))$, this solution can be obtained by solving the equation obtained from (4.2) by using the constraint, or on-shell condition, $k = k'$. This means that the assumption that a function $G(\sigma_1,\sigma_2,r,\boldsymbol{L}^2)$ is the coordinate space representation of a $V(\sigma_1,\sigma_2,\boldsymbol{q},\boldsymbol{p})$ implies restrictions on the off-shell behaviour of V, but no essential restrictions on the on-shell behaviour of V.

The equations we arrive at by using the constraint $k = k'$ are of the type

$$f(x,v) = \int_0^\infty dy g(y,v) J_v^2(xy), \tag{4.9}$$

and

$$f(x,\nu) = \int_0^\infty dy\, y g(y,\nu) J_{\nu-1}(xy) J_{\nu+1}(xy). \tag{4.10}$$

In the equations (4.9) and (4.10) $f(x,\nu)$ is a given function of a real parameter x, and of the complex parameter $\nu \varepsilon D$, where the closed domain D is determined in the process of solving the equations. In order to invert (4.9) and (4.10) we need the following theorems.

Theorem I: If $xf(x,\nu)$ is differentiable in $(0, \infty)$ and if $(xf(x,\nu))' = \frac{d}{dx}(xf(x,\nu))$ belongs to $L^2(0, \infty)$, uniformly with respect to ν within a closed domain to the right of the line $Re(\nu) = -\frac{1}{4}$, the equation

$$f(x,\nu) = \int_0^\infty dy\, y g(y,\nu) J_\nu^2(xy)$$

implies almost everywhere

$$g(x,\nu) = -2\pi \frac{d}{dx} \int_0^\infty \frac{dy}{y} (yf(y,\nu))' \int_0^{xy} du\, u J_\nu(u) Y_\nu(u),$$

and $g(x,\nu)$ also belongs to $L^2(0, \infty)$.

Theorem II: If

$$f(x,\nu) = \int_0^\infty dy\, y g(y,\nu) J_{\nu-1}(xy) J_{\nu+1}(xy),$$

then

$$g(x,\nu) = -\pi \frac{d}{dx} \int_0^\infty \frac{dy}{y} (yf(y,\nu))' \left\{ \int_0^{xy} du\, u [J_{\nu-1}(u) Y_{\nu+1}(u) + J_{\nu+1}(u) Y_{\nu-1}(u)] \right\},$$

the conditions of validity being identical to those given in Theorem I. Here $Y_\nu(u)$ denotes a Bessel function of the second kind. Theorem I was proved in I, and Theorem II is proved in exactly the same manner, so we omit the proof in this paper.

The equations (4.2), (4.3) and (4.6) are now dealt with in exactly the same manner as the equation corresponding to (4.2) which was discussed in detail in I, so we shall only consider eqs. (4.7) and (4.8) in the following. We now assume that the functions $k^2 F_L^{(QT)}(k,k)$ and $k^2 F_L^{(T)}(k,k)$ for non-

negative integral values of L have square integrable derivatives. We then obtain, in accordance with theorems I and II,

$$\left.\begin{aligned} & r(G_2(r,L(L+1)) - 2((L+\tfrac{1}{2})^2 - 1)G_4(r,L(L+1))) \\ & = \frac{(-1)^L}{2\pi}\frac{d}{dr}\int_0^\infty \frac{dk}{k}(k^2F_L^{(QT)}(k,k))' \int_0^{kr} duuJ_{L+\frac{1}{2}}(u)J_{-L-\frac{1}{2}}(u), \end{aligned}\right\} \tag{4.11}$$

and

$$\left.\begin{aligned} & r(G_2(r,(L-1)L) + G_2(r,(L+1)(L+2))) \\ & = \frac{(-1)^L}{4\pi}\frac{d}{dr}\int_0^\infty \frac{dk}{k}(k^2F_L^{(T)}(k,k))' \\ & \left\{\int_0^{kr} duu(J_{L-\frac{1}{2}}(u)J_{-L-\frac{3}{2}}(u) + J_{L+\frac{3}{2}}(u)J_{-L+\frac{1}{2}}(u)).\right\} \end{aligned}\right\} \tag{4.12}$$

(Note that L in (4.11) and (4.12) is a non-negative integer.) In writing down (4.11) and (4.12) we have anticipated a result which should be a consequence of these equations, namely that the functions $G_2(r,L(L+1))$ and $G_4(r,L(L+1))$ actually are entire functions of $L(L+1)$. Let us consider eq. (4.11). The function $J_{L+\frac{1}{2}}(u)J_{-L-\frac{1}{2}}(u)$ is, for fixed positive u, an even entire function of $L+\frac{1}{2}$ and consequently an entire function of $L(L+1)$. We shall then have to prove that $(-1)^LF_L^{(QT)}(k,k)$ can be considered as an entire function of $L(L+1)$. Let us consider the first term in $F_L^{(QT)}(k,k)$, eq. (3.11). Using the symmetry property $P_L(-x) = (-1)^LP_L(x)$ of the Legendre polynominals, we obtain

$$\left.\begin{aligned} (-1)^LF_L^{2(QT)}(k,k) &= \tfrac{2}{3}k^2\int_{-1}^{+1} dx(1-x)P_L(-x)V_2 \\ &+ 3k^2\int_{-1}^{+1} dx\left(\frac{P_{L-1}(-x) - P_{L+1}(-x)}{2L+1}\right)V_2, \end{aligned}\right\} \tag{4.13}$$

where the constraint $k = k'$ is to be used in the expression for V_2. It is known that the function $P_L(-x)$ *qua* function of the complex parameter L, is an even entire function of $L+\frac{1}{2}$, when x has any fixed value such that $-1 \leq x < 1$. The first integral in (4.13) defines therefore an entire function of $L(L+1)$, provided V_2, as a function of x, is continuous in the (open) interval $(-1,1)$, and provided the integral converges uniformly for L within any closed domain. The same reasoning can obviously be carried through for the second term in (4.13), and also for the other terms included in

$(-1)^L F_L^{(QT)}(k,k)$ whence we conclude that $(-1)^L F_L^{(QT)}(k,k)$ can be continued to complex L in such a manner that it becomes an entire function of $L(L+1)$. We may note in passing that this is in general not the continuation to use if one wishes to consider the Schrödinger equation with an angular momentum dependent potential for general complex L, as pointed out in I. The integral on the right hand side of (4.11) then also becomes an entire function of $L(L+1)$, provided the requisite conditions of continuity and uniformity of convergence are satisfied. Let us denote this integral by $I_1(r, L(L+1))$. In exactly the same manner we can prove that the integral on the right hand side of (4.12) can be continued to complex L in such a manner that it becomes an entire function of $L(L+1)$. We denote this integral by $I_2(r, L(L+1))$. We shall now have to prove that $G_2(r, L(L+1))$ and $G_4(r, L(L+1))$ separately are entire functions of $L(L+1)$. Let us for a moment consider the function $G_2(r, L(L+1))$ as a function of $L+\frac{1}{2}$. The equation (4.12) then defines $G_2(r, L(L+1))$ through a difference equation of the form

$$g(z-1) + g(z+1) = f(z), \tag{4.14}$$

where $f(z)$ is an even entire function. It is known[4] that difference equations of this kind have in general solutions; moreover, if $f(z)$ is an integral function of finite order then there exists a particular solution to (4.14) which is an integral function of finite order, as proved in Ref. (4). It is readily seen that this particular solution is even if and only if $f(z)$ is even. As a result of these considerations it is clear that eq. (4.12) defines (apart from arbitrary additive solutions to the homogeneous equation) an even entire function of $L+\frac{1}{2}$; that is, $G_2(r, L(L+1))$ is an entire function of $L(L+1)$. The function $G_4(r, L(L+1))$ defined by (4.11) is consequently regular in the whole finite $L(L+1)$-plane, except possibly for $L(L+1) = \frac{3}{4}$ where a pole can occur. In order that $G_4(r, L(L+1))$ be regular also for $L(L+1) = \frac{3}{4}$, we must have

$$rG_2(r, \tfrac{3}{4}) = I_1(r, \tfrac{3}{4}), \tag{4.15}$$

where, as before, $I_1(r, L(L+1))$ denotes the properly continued integral on the right hand side of (4.11). However, on inserting $L = -\frac{1}{2}$ in eq. (4.12) we obtain

$$2rG_2(r, \tfrac{3}{4}) = I_2(r, -\tfrac{1}{4}). \tag{4.16}$$

It is easily verified that

$$2I_1(r, \tfrac{3}{4}) = I_2(r, -\tfrac{1}{4}), \tag{4.17}$$

whence it follows that $G_4(r, L(L+1))$ is an entire function of $L(L+1)$. We may finally remark that it is not a matter of necessity to obtain the expressions for $G_2(r, L(L+1))$ and $G_4(r, L(L+1))$ separately, since they occur in the radial Schrödinger equations in precisely the combinations given in eq. (4.11) and (4.12). We may also remark that the functions $G_2(r, L(L+1))$ and $G_4(r, L(L+1))$ obtained from eqs. (4.11) and (4.12) cannot be more singular than $o(r^{-\frac{3}{2}})$ near $r = 0$, since we have only considered the class of square integrable functions in the theorems I and II. The conditions which ensure the boundedness property and existence of the absolute moments of the functions G_2 and G_4 can easily be obtained, so we omit these considerations.

We may summarize the previous discussion as follows. We have established the possibility of constructing a local, angular momentum dependent two-nucleon potential starting from the most general momentum space representation of such a potential, and derived the necessary formulae for carrying out such a programme. In order that a given momentum space potential V should correspond exactly to a local, angular momentum dependent potential in coordinate space it is necessary that this V should have an off-shell behaviour which is implicitly defined by eqs. (4.2), (4.3) and eqs. (4.6)–(4.8) once the corresponding functions $G_i(r, L(L+1))$ are obtained by using the on-shell part of these equations. On the energy shell V is only restricted by the differentiability and summability conditions which ensure that the partial wave integral equations have square integrable solutions. These conditions can probably be relaxed so that the resulting solutions can behave like $o(r^{-3})$ near $r = 0$.

5. The scalar boson exchange potential

As an illustration we shall consider the lowest order potential due to the exchange of a scalar boson with mass m and coupling constant g_s. We obtain the following expression for the potential in momentum space:

$$V = V_0 + V_1 i(\boldsymbol{q} \times \boldsymbol{p}) \cdot \boldsymbol{S} + V_4 \sigma_1 \cdot (\boldsymbol{q} \times \boldsymbol{p}) \sigma_2 \cdot (\boldsymbol{q} \times \boldsymbol{p}), \tag{5.1}$$

where

$$V_0 = -\frac{g_s^2 \omega\omega'}{4M\sqrt{EE'}} \left(1 - \frac{\boldsymbol{k} \cdot \boldsymbol{k}'}{\omega\omega'}\right)^2 \frac{1}{\boldsymbol{q}^2 + m^2}, \tag{5.2}$$

$$V_1 = -\frac{g_s^2}{2M\sqrt{EE'}} \left(1 - \frac{\boldsymbol{k} \cdot \boldsymbol{k}'}{\omega\omega'}\right) \frac{1}{\boldsymbol{q}^2 + m^2} \tag{5.3}$$

and

$$V_4 = \frac{g_s^2}{4M\sqrt{EE'\omega\omega'}} \frac{1}{\boldsymbol{q}^2 + m^2}. \tag{5.4}$$

Here we have used the abbreviations

$$\left.\begin{aligned} E &= \sqrt{M^2 + k^2}, \qquad \omega = E + M, \\ E' &= \sqrt{M^2 + k'^2}, \quad \omega' = E' + M, \end{aligned}\right\} \tag{5.5}$$

and $\boldsymbol{q}$ is, as before, the momentum transfer. It is readily seen that the potentials (5.2)–(5.4) cannot, as such, be represented by local, angular momentum dependent potentials in coordinate space. However, since the lowest order potential in principle is fixed only on the energy shell, we may use the on-shell part of the potential directly to obtain the local, angular momentum dependent coordinate space potential. We shall then have to evaluate the functions $F_L^{(O)}, \ldots, F_L^{(T)}$ according to the formulae given in section 3. Let us introduce the notation

$$kF_L^{(i)}(k,k) = f_L^{(i)}(k) + r_L^{(i)}(k), \quad (i = 0,1,SP,QT,T). \tag{5.6}$$

We then obtain, from eqs. (5.2)–5.4),

$$\left.\begin{aligned} f_L^{(0)}(k) &= -\frac{g_s^2 M}{Ek}\left(1 - \frac{m^2}{4M\omega}\right)^2 Q_L(z), \\ r_L^{(0)}(k) &= -\frac{g_s^2 k}{4ME}\left[2\left(1 - \frac{k^2}{2\omega^2}\, z\right)\delta_{L0} - \frac{k^2}{3\omega^2}\,\delta_{L1}\right] \end{aligned}\right\} \tag{5.7}$$

$$\left.\begin{aligned} f_L^{(1)}(k) &= -\frac{g_s^2 k}{E\omega}\left(1 - \frac{m^2}{4M\omega}\right)\left(\frac{Q_{L+1}(z) - Q_{L-1}(z)}{2L+1}\right), \\ r_L^{(1)}(k) &= -\frac{g_s^2 k^3}{2ME\omega^2}\left(\delta_{L0} - \tfrac{1}{3}\,\delta_{L1}\right), \end{aligned}\right\} \tag{5.8}$$

$$\left.\begin{aligned} f_L^{(SP)}(k) &= -\frac{g_s^2 m^2 k}{12ME\omega^2}\left(1 + \frac{m^2}{4k^2}\right) Q_L(z), \\ r_L^{(SP)}(k) &= \frac{g_s^2 k^3}{12ME\omega^2}\left(z\delta_{L0} + \tfrac{1}{3}\,\delta_{L1}\right), \end{aligned}\right\} \tag{5.9}$$

$$\left.\begin{aligned} f_L^{(QT)}(k) &= \frac{g_s^2 m^2 k}{6ME\omega^2}\left(1+\frac{m^2}{4k^2}\right)Q_L(z) \\ &\quad + \frac{g_s^2 k^3 z}{4ME\omega^2}\left(\frac{Q_{L+1}(z)-Q_{L-1}(z)}{2L+1}\right), \\ r_L^{(QT)}(k) &= -\frac{g_s^2 k^3}{2ME\omega^2}\left(\frac{2z+3}{6}\,\delta_{L0}+\frac{5}{18}\,\delta_{L1}\right), \end{aligned}\right\} \tag{5.10}$$

$$\left.\begin{aligned} f_L^{(T)}(k) &= -\frac{g_s^2 k^3}{6ME\omega^2}\left(Q_{L+1}(z)+Q_{L-1}(z)-2zQ_L(z)\right), \\ r_L^{(T)}(k) &= -\frac{g_s^2 k^3}{3ME\omega^2}\,\delta_{L0}. \end{aligned}\right\} \tag{5.11}$$

Here $Q_L(z)$ is the Legendre function, and

$$z = 1+\frac{m^2}{2k^2}. \tag{5.12}$$

We may remark that the formulae (5.8), (5.10) and (5.11) are valid for $L \geq 1$ only, since for $L = 0$ we have to replace the function $Q_{L+1}(z) - Q_{L-1}(z)$ by $Q_1(z) - Q_0(z)$ in accordance with the formulae given in section 3.

Let us then discuss the eq. (5.7). The first term in (5.7) corresponds in the "static" or "adiabatic" limit to the ordinary Yukawa potential. The remainder $r_L^{(0)}(k)$ in (5.7) represents a short range interaction which operates only in the states with $L = 0$ or $L = 1$. Before proceeding further we may remark that it is perhaps unreasonable to pay too much attention to the short range terms $r_L^{(i)}(k)$, since in the states with $L = 0$ and $L = 1$ there are certainly unknown short range interactions which are probably more important than the simple single particle exchange forces of the type considered here. We shall therefore for a moment omit the short range terms $r_L^{(i)}(k)$ from the discussion and consider only the functions $f_L^{(i)}(k)$. We then need the asymptotic expansion[5] of $Q_L(z)$ for large values of k,

$$Q_L(z) = \left(\log\left(\frac{2k}{m}\right) - \gamma - \psi(L+1)\right)(1+O(k^{-2})). \tag{5.13}$$

Here the function $\psi(z)$ is the logarithmic derivative of the gamma function and $\gamma = -\psi(1)$.

It is now readily seen that the function $kf_L^{(0)}(k)$ has a derivative which is square integrable, but also absolutely integrable. We can then interchange the order of differentiation and integration in the inversion formula (Theorem I) given in section 4, and obtain, for integral values of L,

$$G_0(r,L(L+1)) = -\frac{1}{2\pi}\int_0^\infty dk(kf_L^{(0)}(k))'kJ_{L+\frac{1}{2}}(kr)Y_{L+\frac{1}{2}}(kr). \qquad (5.14)$$
$$(L > 1)$$

The integral (5.14) cannot be evaluated in terms of elementary functions, but it can be transformed, by using contour integration, into a form which reveals its relationship to the ordinary Yukawa potential. We have given an example of this in I, and shall not consider the matter further in this paper. The function $kf_L^{(1)}(k)$ has also an absolutely summable derivative, so that we obtain

$$G_1(r,L(L+1)) = -\frac{1}{2\pi}\int_0^\infty dk(kf_L^{(1)}(k))'kJ_{L+\frac{1}{2}}(kr)Y_{L+\frac{1}{2}}(kr). \qquad (5.15)$$
$$(L > 1)$$

Similarly,

$$G_3(r,L(L+1)) = -\frac{1}{2\pi}\int_0^\infty dk(kf_L^{(SP)}(k))'kJ_{L+\frac{1}{2}}(kr)Y_{L+\frac{1}{2}}(kr). \qquad (5.16)$$
$$(L > 1)$$

We may remark that the functions G_0, G_1 and G_3 defined by the eqs. (5.14)–(5.16) are less singular than r^{-1} near $r = 0$.

For the remaining functions, $f_L^{(QT)}(k)$ and $f_L^{(T)}(k)$, the situation is different. We observe that these functions tend to definite (non-zero) limits when k tends to infinity:

$$\lim_{k\to\infty} f_L^{(QT)}(k) = \frac{g_s^2}{4M}\left(\frac{\psi(L-1)-\psi(L+1)}{2L+1}\right), \qquad (5.17)$$

$$\lim_{k\to\infty} f_L^{(T)}(k) = -\frac{g_s^2}{6M}(2\psi(L)-\psi(L+1)-\psi(L-1)). \qquad (5.18)$$

The functions $kf_L^{(QT)}(k)$ and $kf_L^{(T)}(k)$ do therefore not have square integrable derivatives, and the inversion theorems are not immediately applicable in these cases. This difficulty can readily be overcome. Suppose now that the principal part near $r = 0$ of the functions

$$G_2(r, L(L+1)) - \tfrac{1}{2}(2L-1)(2L+3)G_4(r, L(L+1))$$

and

$$G_2(r, (L-1)L) + G_2(r, (L+1)(L+2))$$

is of the form

$$g_{24}(L) r^{-\lambda-1}, \qquad g_{22}(L) r^{-\lambda-1} \tag{5.19}$$

respectively, where $0 < \lambda < 2$ and g_{22}, g_{24} are L-dependent constants. Recalling the formula[6]

$$\left.\begin{aligned} \int_0^\infty dr r^{-\lambda} J_\mu(kr) J_\nu(kr) &= C(\lambda,\mu,\nu) k^{\lambda-1}, \\ (\mathrm{Re}(\mu+\nu+1) &> \mathrm{Re}(\lambda) > 0) \end{aligned}\right\} \tag{5.20}$$

where

$$\left.\begin{aligned} C(\lambda,\mu,\nu) &= \\ \frac{2^{-\lambda}\Gamma(\lambda)\Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu - \tfrac{1}{2}\lambda + \tfrac{1}{2})}{\Gamma(\tfrac{1}{2}\lambda + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu + \tfrac{1}{2})\Gamma(\tfrac{1}{2}\lambda + \tfrac{1}{2}\mu + \tfrac{1}{2}\nu + \tfrac{1}{2})\Gamma(\tfrac{1}{2}\lambda + \tfrac{1}{2}\mu - \tfrac{1}{2}\nu + \tfrac{1}{2})}&, \end{aligned}\right\} \tag{5.21}$$

we observe, from eqs. (4.7) and (4.8), that λ in (5.19) must equal unity, and obtain the relations

$$4\pi^2 g_{24}(L) C(1, L+\tfrac{1}{2}, L+\tfrac{1}{2}) = f_L^{(QT)}(\infty), \tag{5.22}$$

$$-4\pi^2 g_{22}(L) C(1, L-\tfrac{1}{2}, L+\tfrac{3}{2}) = f_L^{(T)}(\infty), \tag{5.23}$$

where $f_L^{(QT)}(\infty)$, $f_L^{(T)}(\infty)$ denote the limits (5.17) and (5.18), respectively. We then obtain (for $L > 1$)

$$\left.\begin{aligned} &G_2(r, L(L+1)) - 2((L+\tfrac{1}{2})^2 - 1) G_4(r, L(L+1)) = \\ &\frac{g_{24}(L)}{r^2} - \frac{1}{2\pi}\int_0^\infty dk (k(f_L^{(QT)}(k) - f_L^{(QT)}(\infty)))' k J_{L+\frac{1}{2}}(kr) Y_{L+\frac{1}{2}}(kr), \end{aligned}\right\} \tag{5.24}$$

and

$$\left.\begin{aligned} &G_2(r, (L-1)L) + G_2(r, L+1)(L+2)) = \\ &\frac{g_{22}(L)}{r^2} + \frac{1}{4\pi}\int_0^\infty dk (k(f_L^{(T)}(k) - f_L^{(T)}(\infty)))' k \\ &\{J_{L-\frac{1}{2}}(kr) Y_{L+\frac{3}{2}}(kr) + J_{L+\frac{3}{2}}(kr) Y_{L-\frac{1}{2}}(kr)\}. \end{aligned}\right\} \tag{5.25}$$

As we mentioned before, it is not necessary to obtain the expressions for G_2 and G_4 separately, since these functions occur in the radial Schrödinger equations in precisely the combinations given in (5.24) and (5.25).

Let us now return to the short range terms $r_L^{i)}(k)$. It is readily seen that these terms tend to non-zero limits $r_L^{(i)}(\infty)$ when k tends to infinity. By using the same arguments as above, we conclude that these terms in coordinate space correspond to potentials which behave like $O(r^{-2})$ near $r = 0$.

We may summarize the discussion as follows. The scalar boson exchange potential (5.1) can be represented by a local, angular momentum dependent potential G in coordinate space, which contains a central .., spin-orbit, spin-spin .., quadratic spin orbit .. and tensor force potential. The three first mentioned potentials behave like $o(r^{-1})$ near $r = 0$, whereas the quadratic spin orbit .. and tensor force potentials behave like $O(r^{-2})$ near $r = 0$. In addition, there appear short range terms which operate only in the states with $L = 0$ and $L = 1$, respectively, and which behave like $O(r^{-2})$ near $r = 0$.

Let us now compare this result to the one which is obtained by using expansions with respect to $\boldsymbol{p}^2/M^2$. In the adiabatic limit[7)] the spin-bilinear terms disappear completely. The central potential becomes the ordinary Yukawa potential $Y = \dfrac{e^{-mr}}{r}$, and the spin-orbit force becomes $\dfrac{1}{r}\dfrac{d}{dr}Y(mr)$, which behaves like $O(r^{-3})$ near $r = 0$. In the next approximation, keeping terms of the order $\boldsymbol{p}^2/M^2$, one obtains[8)] a quadratically momentum dependent central potential, which in coordinate space has the form $Y(mr) - \dfrac{\hat{p}^2}{M^2}Y(mr)$, where $\hat{p}$ is the differential operator $-\dfrac{i}{2}\left(\overrightarrow{\partial} - \overleftarrow{\partial}\right)$. The resulting "effective" central potential is therefore a linearly energy dependent function which behaves like $O(r^{-3})$ near $r = 0$. The spin orbit potential is in this approximation the same as in the adiabatic limit. Also, the spin-bilinear terms are absent in this approximation. We may therefore conclude that the approximations of the above mentioned type give a both qualitatively and quantitatively misleading picture of the scalar boson exchange potential.

6. Concluding remarks

In the foregoing sections we have considered the more or less formal problem of obtaining a local, angular momentum dependent potential from a given potential in the momentum representation. The formalism outlined here might be considered complicated, but this lies in the nature of the

problem and cannot be avoided. We may note that if one wishes to solve the two-nucleon scattering problem with a non-local potential in the momentum representation by using the Lippmann-Schwinger equation, then the matrix elements evaluated in section 3 are necessary ingredients in a calculation of this kind. We have shown that a potential V in the momentum representation becomes local and angular momentum dependent in coordinate space if V has a particular off-shell behaviour, which is implicitly defined by the partial wave integral equations considered in this paper. We mentioned in the introduction that in a perturbative definition of a potential it is always possible to determine the off-shell behaviour so that the coordinate space potential becomes local and angular momentum dependent. We can in fact add an arbitrary term, vanishing on the energy shell, to the lowest order potential, provided this is compensated by adding the proper corrections to the higher order potentials. This procedure can then be repeated for the potential of next order, and so on.

It is thus possible, within a perturbative definition of the potential, to get a potential with any (reasonable) off-shell behaviour, and, in particular, the off-shell behaviour which yields a local, angular momentum dependent potential in coordinate space.

Among potentials of this kind we may mention the various one-boson-exchange potentials, which have been used extensively in numerical calculations in nucleon-nucleon (N-N) scattering. We shall not enter upon a discussion of the domain of applicability of such potentials in this paper; we merely recognize the fact that the vast majority of potentials which have been considered in N-N scattering are based more or less directly on the use of perturbation theory expansions for the S-matrix from field theory. Besides the approximations of a "physical" nature involved in calculations with such potentials, one has used approximations involving expansions with respect to the inverse of the nucleon mass, in order to obtain local or "almost local" potentials.

In I we have investigated the validity of such approximations by evaluating the phase shifts for the case of scalar particles interacting through a single particle exchange potential. It was shown that the "adiabatic" approximation can lead to quite inaccurate results for this case. The example considered in section 5 of this paper shows clearly that the approximation involving an expansion in $\boldsymbol{p}^2/M^2$ is entirely misleading.

It is of course very natural to make approximations which lead to local potentials, in order to obtain manageable equations. As we have demonstrated, this can be achieved, whithout using $\boldsymbol{p}^2/M^2$-expansions, by taking

advantage of the ambiguities inherent in any S-matrix definition of the potential, and this leads to the concept of a local, angular momentum dependent potential.

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References

1) C. Cronström, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **35** (13) (1967).
2) J. Goto and S. Machida, Progr. Theor. Phys., **25**, 64 (1961).
3) S. Okubo and R. E. Marshak, Ann. Phys. (U.S.A.) **4**, 166 (1958).
4) E. W. Barnes, Proc. Lond. Math. Soc. (2), II, 438 (1904).
5) A. Erdélyi (Ed.), Higher Transcendental Functions, Vol. I, McGraw-Hill Book Company, Inc., New York (1953).
6) G. N. Watson, Theory of Bessel Functions, (2nd ed.), Cambridge University Press (1944).
7) R. A. Bryan, C. R. Dismukes, and W. Ramsay, Nucl. Phys., **45**, 353 (1963).
8) N. Hoshizaki, I. Lin and S. Machida, Progr. Theor. Phys. **26**, 680 (1961).

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